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ON STRONGLY (p, h)-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new class of convex functions which is called strongly (p, h)-convex functions. We show that this class includes several other new classes of convex functions. We also establish some new results for Hermite-Hadamard type inequalities via strongly (p, h)-convex functions. Some special cases which can be deduced from our main results are also discussed.

Keywords: convex functions, p-convex functions, strongly (p, h)-convex functions, Hermite-Hadamard inequalities.

AMS Subject Classification: 26D15, 26A51.

1. Introduction and preliminaries

In recent years many researchers have generalized the classical concepts of convex sets and convex functions in different directions using novel approaches, see [1, 2, 3, 4, 6, 7, 8, 21]. Theory of convexity has many applications in different fields of pure and applied sciences. It has a strong relationship with theory of inequalities. Consequently many inequalities have been obtained via convex functions, see [4, 5, 6, 7, 8, 9, 11, 12, 10, 13, 14, 15, 16, 17, 18, 19, 20]. A significant class of convex functions is that of strongly convex functions which was introduced in [22]. Strongly convex functions are being used to construct some iterative methods for solving variational inequalities and related optimization problems. The Hermite-Hadamard inequality for strongly convex functions was obtained in [10]. For various applications of strongly convex functions in variational inequalities and optimization, see [1, 2, 10, 14, 15, 17, 18, 19, 22] and the references therein. The aim of this paper is to define a new class of convex functions, which is called the strongly (p, h)-convex function. We show that this class unifies several other new and known classes of convex functions. We derive some new estimates of Hermite-Hadamard type inequalities via strongly (p, h)-convex functions. Several new and known special cases which can be deduced from our main results are also discussed. First of all, we recall some previously known concepts.

Definition 1.1 ([23]). *An interval I is said to be a p-convex set if*

$$M_p(x, y; t) = [tx^p + (1-t)y^p]^{\frac{1}{p}} \in I$$

for all $x, y \in I, t \in [0, 1]$, where $p = 2k + 1$ or $p = \frac{n}{m}, n = 2r + 1, m = 2t + 1$ and $k, r, t \in \mathbb{N}$.

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Definition 1.2 ([23]). *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be p-convex function or belongs to the class $PC(I)$, if*

$$f(M_p(x, y; t)) \le tf(x) + (1 - t)f(y),\ \forall x, y \in I, t \in [0, 1].$$

It is very much obvious that for $p = 1$ Definition 1.2 reduces to the definition for classical convex functions.
Note that for $p = -1$, we have the definition of harmonically convex functions:

Definition 1.3 ([8]). *A function $f : H \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically convex function, if*

$$f\left(\frac{xy}{(1 - t)x + ty}\right) \le tf(x) + (1 - t)f(y),\ \forall x, y \in I, t \in [0, 1].$$

Also note that for $t = \frac{1}{2}$ in Definition 1.2, we have Jensen p-convex functions or mid p-convex functions:

$$f(M_p(x, y; 1/2)) \le \frac{f(x) + f(y)}{2},\ \forall x, y \in I, t \in [0, 1].$$

It have shown that a minimum of a differentiable harmonic convex functions can be characterized by a class of variational inequalities, which is called harmonic variational inequalities, see [15]. Fang et al. [7] introduced a new class of convex functions, which is called as (p, h)-convex functions.

Definition 1.4. *Let $h : J \to \mathbb{R}$ be a non-negative and $h \neq 0$. A function $f : I \to \mathbb{R}$ is said to be (p, h)-convex function, if f is non-negative and*

$$f(M_p(x, y; t)) \le h(t)f(x) + h(1 - t)f(y),\ \forall x, y \in I, t \in (0, 1).$$

2. New notions

In this section, we introduce some new classes of convex functions. First of all, we introduce the class of strongly (p, h)-convex functions.

Definition 2.1. *Let $h : J \to \mathbb{R}$ be a non-negative and $h \neq 0$. A function $f : I \to \mathbb{R}$ is said to be strongly (p, h)-convex function with modulus $\mu > 0$, if*

$$f(M_p(x, y; t)) \le h(t)f(x) + h(1 - t)f(y) - \mu t(1 - t)(y^p - x^p)^2,\ \forall x, y \in I, t \in (0, 1).$$

Note that if $\mu = 0$ in Definition 2.1, then, we have Definition 1.4.
I. If $h(t) = t$ in Definition 2.1, then, we have definition of strongly p-convex functions, which appears to be new one.

Definition 2.2. *A function $f : I \to \mathbb{R}$ is said to be strongly p-convex function with modulus $\mu > 0$, if*

$$f(M_p(x, y; t)) \le tf(x) + (1 - t)f(y) - \mu t(1 - t)(y^p - x^p)^2,\ \forall x, y \in I, t \in (0, 1).$$

II. If $h(t) = t^s$ in Definition 2.1, then, we have definition of Breckner type of strongly (p, s)-convex functions, which is a new one.

Definition 2.3. *A function $f : I \to \mathbb{R}$ is said to be Breckner type of strongly (p, s)-convex function with modulus $\mu > 0$, if*

$$f(M_p(x, y; t)) \le t^s f(x) + (1 - t)^s f(y) - \mu t(1 - t)(y^p - x^p)^2,$$
$$\forall x, y \in I, t \in (0, 1), s \in [0, 1].$$

III. If $h(t) = t^{-s}$ in Definition 2.1, then, we have definition of Godunova-Levin type of strongly (p, s)-convex functions.

Definition 2.4. *A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin type of strongly (p, s)-convex function with modulus $\mu > 0$, if*

$$f(M_p(x, y; t)) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y) - \mu t(1-t)(y^p - x^p)^2,$$
$$\forall x, y \in I, t \in (0, 1), s \in [0, 1].$$

IV. If $h(t) = t^{-1}$ in Definition 2.1, then, we have definition of Godunova-Levin type of strongly p-convex functions, which appears to be a new one.

Definition 2.5. *A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin type of strongly p-convex function with modulus $\mu > 0$, if*

$$f(M_p(x, y; t)) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y) - \mu t(1-t)(y^p - x^p)^2,$$
$$\forall x, y \in I, t \in (0, 1).$$

V. If $h(t) = 1$ in Definition 2.1, then, we have definition of strongly (p, P)-convex functions.

Definition 2.6. *A function $f : I \to \mathbb{R}$ is said to be strongly (p, P)-convex function with modulus $\mu > 0$, if*

$$f(M_p(x, y; t)) \leq f(x) + f(y) - \mu t(1-t)(y^p - x^p)^2, \ \forall x, y \in I, t \in (0, 1).$$

Remark 2.1. *We would like to remark here that, for $p = 1$ in Definition 2.1, we have definition for strongly h-convex functions, see [1]. And for $p = -1$ in Definition 2.1, we have definition for strongly harmonic h-convex functions, which also appears to be new one.*

Definition 2.7. *A function $f : H \setminus \{0\} \to \mathbb{R}$ is said to be strongly harmonic h-convex function with modulus $\mu > 0$, if*

$$f\left(\frac{xy}{(1-t)x + ty}\right) \leq h(t) f(x) + h(1-t) f(y) - \mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2,$$
$$\forall x, y \in I, t \in (0, 1).$$

Note that, if $h(t) = t$ in Definition 2.7, we have definition for strongly harmonic convex functions, see [19]. If $h(t) = t^s$, then, we have definition for Breckner type of strongly harmonic s-convex functions.

Definition 2.8. *A function $f : H \setminus \{0\} \to \mathbb{R}$ is said to be Breckner type of strongly harmonic s-convex function, $s \in [0, 1]$ with modulus $\mu > 0$, if*

$$f\left(\frac{xy}{(1-t)x + ty}\right) \leq t^s f(x) + (1-t)^s f(y) - \mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2,$$
$$\forall x, y \in I, t \in (0, 1).$$

If $h(t) = t^{-s}$, then, we have definition for Godunova-Levin type of strongly harmonic s-convex functions.

Definition 2.9. *A function $f : H \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin type of strongly harmonic s-convex function, $s \in [0, 1]$ with modulus $\mu > 0$, if*

$$f\left(\frac{xy}{(1-t)x + ty}\right) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y) - \mu t(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2,$$
$$\forall x, y \in I, t \in (0, 1).$$

If $h(t) = t^{-1}$, then, we have definition for Godunova-Levin type of strongly harmonic convex functions.

Definition 2.10. *A function $f : H \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin type of strongly harmonic convex function with modulus $\mu > 0$, if*

$$f\left(\frac{xy}{(1-t)x+ty}\right) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y) - \mu t(1-t)\Big(\frac{1}{y} - \frac{1}{x}\Big)^2, \quad \forall x, y \in I, t \in (0,1).$$

If $h(t) = 1$, then, we have definition for strongly harmonic P-convex functions.

Definition 2.11. *A function $f : H \setminus \{0\} \to \mathbb{R}$ is said to be strongly harmonic P-convex function with modulus $\mu > 0$, if*

$$f\left(\frac{xy}{(1-t)x+ty}\right) \leq f(x) + f(y) - \mu t(1-t)\Big(\frac{1}{y} - \frac{1}{x}\Big)^2, \ \forall x, y \in I, t \in (0,1).$$

We would like to emphasize that for appropriate and suitable choices of the arbitrary function $h(.)$ and the parameter p, one can obtain a wide class of strongly convex functions as special cases from Definition 2.1. This shows that Definition 2.1 is a general unifying one.

3. Hermite-Hadamard type inequalities via strongly (p, h)-convex functions

In this section, we derive our main results and discuss some special cases.

Theorem 3.1. *Let $f : I \to \mathbb{R}$ be strongly (p, h)-convex function with modulus $\mu > 0$. Then, for $h\big(\frac{1}{2}\big) \neq 0$, we have*

$$\frac{1}{2h\big(\frac{1}{2}\big)}\left[f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) + \frac{\mu}{12}(b^p - a^p)^2\right]$$
$$\leq \frac{p}{b^p - a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x \leq (f(a) + f(b))\int_0^1 h(t)\mathrm{d}t - \frac{\mu}{6}(b^p - a^p)^2.$$

Proof. For $t \in (0,1)$, let $x = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ and $y = [(1-t)a^p + tb^p]^{\frac{1}{p}}$. Using strongly (p, h)-convexity of f, we get

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) = f\left(\left[\frac{x^p + y^p}{2}\right]^{\frac{1}{p}}\right)$$
$$\leq h\Big(\frac{1}{2}\Big)\left[f([ta^p + (1-t)b^p]^{\frac{1}{p}}) + f([(1-t)a^p + tb^p]^{\frac{1}{p}})\right] - \frac{\mu}{4}(1-2t)^2(b^p - a^p)^2.$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{2h\big(\frac{1}{2}\big)}\left[f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) + \frac{\mu}{12}(b^p - a^p)^2\right] \leq \frac{p}{b^p - a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x. \tag{1}$$

Also

$$f([ta^p + (1-t)b^p]^{\frac{1}{p}}) \leq h(t)f(a) + h(1-t)f(b) - \mu t(1-t)(b^p - a^p)^2.$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x \leq (f(a)+f(b))\int_0^1 h(t)\mathrm{d}t - \frac{\mu}{6}(b^p-a^p)^2. \tag{2}$$

Summation of (1) and (2) completes the proof. □

We now discuss some special cases of Theorem 3.1.

I. If $p=1$, in Theorem 3.1, we have Theorem 4.1 [1]. If $p=-1$, in Theorem 3.1, we have:

Theorem 3.2. *Let $f: I \to \mathbb{R}$ be strongly harmonic h-convex function with modulus $\mu > 0$.then, for $h\left(\frac{1}{2}\right) \neq 0$, we have*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{2ab}{a+b}\right)+\frac{\mu(b-a)^2}{12a^2b^2}\right]$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq (f(a)+f(b))\int_0^1 h(t)\mathrm{d}t - \frac{\mu(b-a)^2}{6a^2b^2}.$$

If $h(t)=t$ and $p=1$, in Theorem 3.1, we have:

Theorem 3.3. *Let $f: I \to \mathbb{R}$ be strongly convex function with modulus $\mu > 0$ then, we have*

$$f\left(\frac{a+b}{2}\right)+\frac{\mu(b-a)^2}{12} \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} - \frac{\mu(b-a)^2}{6}.$$

When $h(t)=t$ and $p=-1$, we have:

Theorem 3.4 ([19]). *Let $f: I \to \mathbb{R}$ be strongly harmonic convex function with modulus $\mu > 0$ then, we have*

$$f\left(\frac{2ab}{a+b}\right)+\frac{\mu(b-a)^2}{12a^2b^2} \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2} - \frac{\mu(b-a)^2}{6a^2b^2}.$$

II. If $h(t)=t$, in Theorem 3.1, we have following new result for strongly p-convex functions.

Corollary 3.1. *Let $f: I \to \mathbb{R}$ be strongly p-convex function with modulus $\mu > 0$, then, we have*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{\mu}{12}(b^p-a^p)^2$$
$$\leq \frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} - \frac{\mu}{6}(b^p-a^p)^2.$$

III. If $h(t)=t^s$, in Theorem 3.1, we have following new result for Breckner type of strongly (p,s)-convex functions.

Corollary 3.2. *Let $f: I \to \mathbb{R}$ be Breckner type of strongly (p,s)-convex function with modulus $\mu > 0$, then, we have*

$$2^{s-1}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{\mu}{12}(b^p-a^p)^2\right]$$
$$\leq \frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{s+1} - \frac{\mu}{6}(b^p-a^p)^2.$$

IV. If $h(t) = t^{-s}$, in Theorem 3.1, we have following new result for Godunova-Levin type of strongly (p, s)-convex functions.

Corollary 3.3. *Let $f : I \to \mathbb{R}$ be Godunova-Levin type of strongly (p, s)-convex function with modulus $\mu > 0$, then, we have*

$$\frac{1}{2^{1+s}}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{\mu}{12}(b^p-a^p)^2\right]$$
$$\leq \frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{1-s}-\frac{\mu}{6}(b^p-a^p)^2.$$

V. If $h(t) = 1$, in Theorem 3.1, we have following new result for strongly (p, P)-convex functions.

Corollary 3.4. *Let $f : I \to \mathbb{R}$ be strongly (p, P)-convex function with modulus $\mu > 0$, then, we have*

$$\frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{\mu}{12}(b^p-a^p)^2\right]$$
$$\leq \frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x \leq (f(a)+f(b))-\frac{\mu}{6}(b^p-a^p)^2.$$

VI. If $h(t) = t^s$ and $p = -1$, in Theorem 3.1, we have following new result for Breckner type of strongly harmonic s-convex functions.

Corollary 3.5. *Let $f : I \to \mathbb{R}$ be Breckner type of strongly harmonic s-convex function with modulus $\mu > 0$, then, we have*

$$2^{s-1}\left[f\left(\frac{2ab}{a+b}\right)+\frac{\mu(b-a)^2}{12a^2b^2}\right] \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{s+1}-\frac{\mu(b-a)^2}{6a^2b^2}.$$

VII. If $h(t) = t^{-s}$ and $p = -1$, in Theorem 3.1, we have following new result for Godunova-Levin type of strongly harmonic s-convex functions.

Corollary 3.6. *Let $f : I \to \mathbb{R}$ be Godunova-Levin type of strongly harmonic s-convex function with modulus $\mu > 0$, then, we have*

$$\frac{1}{2^{1+s}}\left[f\left(\frac{2ab}{a+b}\right)+\frac{\mu(b-a)^2}{12a^2b^2}\right] \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{1-s}-\frac{\mu(b-a)^2}{6a^2b^2}.$$

VIII. If $h(t) = 1$ and $p = -1$, in Theorem 3.1, we have following new result for strongly harmonic P-convex functions.

Corollary 3.7. *Let $f : I \to \mathbb{R}$ be strongly harmonic P-convex function with modulus $\mu > 0$, then, we have*

$$\frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{\mu(b-a)^2}{12a^2b^2}\right] \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq (f(a)+f(b))-\frac{\mu(b-a)^2}{6a^2b^2}.$$

Theorem 3.5. *Let $f, g : I \to \mathbb{R}$ be non-negative two strongly (p,h)-convex functions, then,*

$$\begin{aligned}
&\frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)g(x)\mathrm{d}x \\
&\le M(a,b)\int_0^1 h_1(t)h_2(t)\mathrm{d}t + N(a,b)\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t \\
&\qquad - \frac{\mu}{12}P(a,b)(b^p - a^p)^2 + \frac{\mu^2}{30}(b^p - a^p)^2.
\end{aligned}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b), \tag{3}$$

$$N(a,b) = f(a)g(b) + f(b)g(a), \tag{4}$$

and

$$P(a,b) = f(a) + g(a) + f(b) + g(b). \tag{5}$$

Proof. Since f and g are strongly (p,h)-convex function, then

$$\begin{aligned}
&f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) g\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) \\
&\le [h_1(t)f(a) + h_1(1-t)f(b) - \mu t(1-t)(b^p - a^p)] \\
&\qquad \times [h_2(t)g(a) + h_2(1-t)g(b) - \mu t(1-t)(b^p - a^p)] \\
&= h_1(t)h_2(t)f(a)g(a) + h_1(t)h_2(1-t)f(a)g(b) - \mu t^2(1-t)f(a)(b^p - a^p)^2 \\
&\quad + h_1(t)h_2(1-t)f(b)g(a) \\
&\quad + h_1(1-t)h_2(1-t)f(b)g(b) - \mu t(1-t)^2 f(b)(b^p - a^p)^2 \\
&\quad - \mu t^2(1-t)g(a)(b^p - a^p)^2 - \mu t(1-t)^2 g(b)(b^p - a^p)^2 + \mu^2 t^2(1-t)^2(b^p - a^p)^2.
\end{aligned}$$

Integrating above inequality with respect to t on the interval $[0,1]$, we have

$$\begin{aligned}
&\frac{p}{b^p - a^p}\int_a^b x^{p-1} f(x)g(x)\mathrm{d}x \\
&\le M(a,b)\int_0^1 h_1(t)h_2(t)\mathrm{d}t + N(a,b)\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t \\
&\qquad - \frac{\mu}{12}\{f(a) + g(a) + f(b) + g(b)\}(b^p - a^p)^2 + \frac{\mu^2}{30}(b^p - a^p)^2.
\end{aligned}$$

This completes the proof. □

4. Conclusion

In this paper, we have introduced and studied a general and unified class of strongly convex functions involving an arbitrary function h and a parameter p. It is shown that a wide class of convex functions and their forms can be obtained as special cases. Several new Hermite-Hadamard type inequalities are established as applications of our results, we have discussed some special cases.

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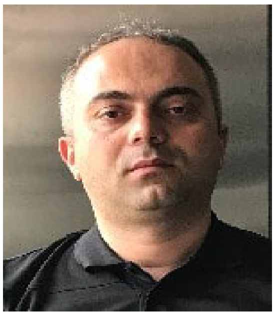

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